# AN INVERTIBLE NON-POLYHEDRAL DIAGRAM

BY
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ABSTRACT
A necessary condition for a diagram to be a Schlegel diagram of some convex polytope is that it be invertible, that is, it can be realized with any facet as the outside facet. In this paper we prove that it is not a sufficient condition by constructing a 3-diagram that is invertible but is not isomorphic to a Schlegel diagram of any 4-polytope.

## 1. Introduction

One way of representing the boundary complex of a convex polytope is by means of a *Schlegel diagram*. A Schlegel diagram is constructed by taking a point close to the centroid of a facet and then projecting the boundary of the polytope from that point onto the facet. The result is that the images of the faces of the polytope that do not lie in the chosen facet are projected onto a collection $C$ of polytopes with the following properties:

(i) The intersection of any two polytopes in $C$ is a polytope in $C$ (possibly empty).

(ii) A face of any polytope in $C$ is in $C$.

(iii) The union of the polytopes in $C$ is a convex polytope $P$.

(iv) The intersection of the boundary of $P$ with any polytope in $C$ is a polytope in $C$ (possibly empty).

All of the polytopes in $C$, together with the union of the polytopes in $C$, will be called the *faces* of $C$. The union of the polytopes in $C$ will be called the *outside facet* of $C$. A face of dimension $d$ will be called a $d$*-face*. A face of highest dimension $C$ will be called a *facet* of $C$.

Any finite collection of convex polytopes satisfying the above four properties will be called a *diagram*. A diagram of dimension $d$ will be called a $d$*-diagram*.

† Research supported by NSF Grant #NSF MCS-07466.
Received November 22, 1977 and in revised form January 17, 1980

We say that two diagrams are *isomorphic* provided there is a one-to-one function of the set of faces of one onto the set of faces of the other such that dimension and incidences are preserved. We say that a $d$-diagram is *polyhedral,* provided it is isomorphic to the Schlegel diagram of some $(d + 1)$-polytope.

It follows from Steinitz' Theorem [3] that every 2-diagram is polyhedral. Examples are known (see [2], Ch. 11 and [1]) of 3-diagrams that are not polyhedral.

Necessary and sufficient conditions are not known for a 3-diagram to be polyhedral, but some necessary conditions are known. One of these conditions is that the diagram must be invertible. This means that for each facet $F$ of the diagram there exists an isomorphic diagram with the isomorphism taking $F$ onto the outside facet. One can think of this as turning the diagram "inside out".

Until now, all examples of non-polyhedral diagrams have failed to be invertible. We shall construct an invertible non-polyhedral diagram.

We shall use "con" to denote the convex hull of a set, however, when the set is a set of vertices we shall simply list the vertices, thus *abcd* would denote $\mathrm{con}(a, b, c, d)$.

If $F$ is a facet of a 3-polytope $P$, we say that a 3-polytope $P'$ is obtained from $P$ by adding a pyramidal cap over $F$, if $P' = \mathrm{con}(x, P)$ where $x$ is a point outside $P$ very close to the centroid of $F$.

## 2. Construction of the diagram

We begin with a right triangular prism with parallel bases. We rotate the top triangle slightly about a vertical axis so that the convex hull of the top and bottom triangles is an octahedron (Fig. 1). We let the vertices of the octahedron be $a, b, \cdots, f$ as in the figure.

Instead of the faces of the octahedron, we shall use the faces *afd*, *afc*, *cef*, *ceb*, *adb*, *dbe*, *abc* and *def* as 2-dimensional faces in our diagram. These form a sphere which we shall call $C$.

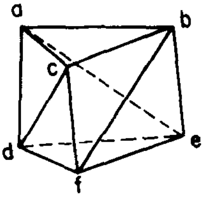


Fig. 1.

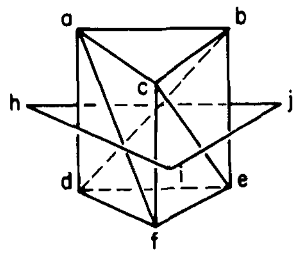


Fig. 2.

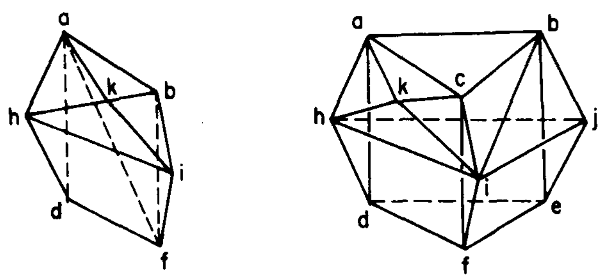


Fig. 3.

We take a triangle $hij$ lying in a plane halfway between and parallel to the planes of $abc$ and $def$, with the triangle lying outside of $C$, with $hi$, $ij$ and $jh$ parallel to $df$, $fe$ and $ed$, respectively (Fig. 2).

Consider the 3-polytope $P = acdfhi$. We choose a point $k$ near the midpoint of edge $hc$, outside $P$, but close enough to $P$ so that the polytope $F_1 = acdfhik$ is as shown in Fig. 3. Similarly we may choose points $l$ and $m$ so that the polytopes $F_2 = cbefijl$ and $F_3 = abdehim$ are of the same combinatorial type as $F_1$. These polytopes will be facets of our diagram. If the points $k$, $l$ and $m$ are chosen close enough to the convex hull of $\{a, b, c, d, e, f, h, i, j\}$ then the 2-dimensional faces of the outside portion of the set we have constructed will form the boundary of a convex polytope which we shall call $P'$.

We choose a point $g$ inside $C$ such that the segments from $g$ to each of the vertices $a, b, \cdots, f$ lie inside $C$. We add to our complex all tetrahedra of the form $\operatorname{con}(g, X)$ where $X$ is a 2-dimensional face of $C$.

We have now filled $P'$ with 3-dimensional polytopes — eight simplices and the polytopes $F_1$, $F_2$ and $F_3$. We complete our construction by choosing a facet $H$ of $P'$ (it doesn't matter which one) and applying a projective transformation to $P'$ so that from some point $n$, every facet of $P'$ except the facet $H$ will be visible.

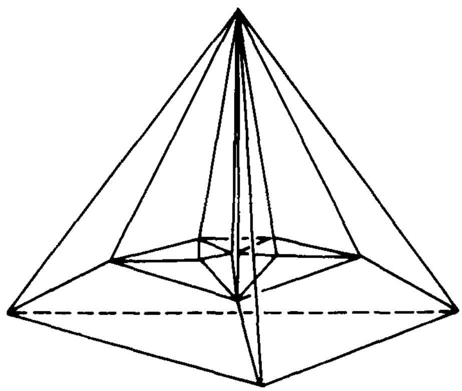

Fig. 4.

The projective transformation will take the complex that we have constructed onto a complex of the same combinatorial type. To this projective image of our complex we add all polytopes of the form con$(n, X)$ where $X$ is a facet of $P'$ visible from $n$ (Fig. 4). This completes the construction of the diagram.

Now let us see why the diagram is not polyhedral. Suppose some 4-polytope $Q$ was isomorphic to the diagram. Let all vertices, faces, subcomplexes, etc. be labeled in $Q$ the same as in the diagram.

The subcomplex $C$ is a 2-sphere that separates the boundary of $Q$ into two regions, one containing the vertex $g$ and all faces containing $g$. This region we will call the *inside* of $C$.

If we remove the vertex $g$ and take the convex hull of the remaining vertices, the net effect is that the faces inside $C$ are removed and the inside of $C$ is filled with polytopes that are convex hulls of vertices of $C$. We shall refer to the region inside $C$ as the "hole".

We shall now see that it is impossible to fill up the hole. The hole cannot be filled by a single facet because $F_1$ would meet this new facet on two 2-faces (namely $adf$ and $afc$).

Suppose the hole is filled in such a way that an edge is introduced inside $C$, such an edge must join one of the pairs of vertices $ae$, $bf$ or $cd$ because all other pairs are joined on $C$. It is impossible for $ae$ to be an edge because such an edge would meet $F_3$ at two points but would not be an edge of $F_3$. Similarly, the edges $bf$ and $cd$ are impossible. We conclude that there will not be any edges introduced inside $C$.

Since there is more than one new facet inside $C$ there will be 2-faces lying inside $C$. Furthermore if there are no edges lying inside $C$, then any 2-faces inside $C$ will span a circuit on $C$. It is clear that such a circuit cannot have any diagonals on $C$, thus the only kind of circuit that could be spanned by a 2-face is a circuit that separates one vertex of $\{a, b, c\}$ from one vertex of $\{d, e, f\}$. If one such circuit is spanned by a 2-cell then neither of the other two circuits can be spanned by a 2-cell without the 2-cells interse cting, thus we conclude that there is exactly one 2-cell inside $C$ and there are exactly two new facets inside $C$. We shall look at the case where the circuit is *abef* — the same argument works in the other two cases.

If the circuit is *abef* then one of the two new facets is a pyramid over a quadrilateral with vertices *abefc*. This cannot be, however, because this facet meets $F_2$ on two 2-faces, namely *cef* and *cbe*.

## 3. Inverting the diagram

In the construction of the diagram, the choice of the facet $H$ is arbitrary, thus the diagram can be realized with any of the facets meeting $n$ serving as the outside facet.

Now we show how to realize the diagram wih $F_1$ as the outside facet. We begin wih a right triangular prism with parallel bases, with vertices $a$, $b$ and $c$ on the top face, and $d$, $e$ and $f$ on the bottom face. We shrink the edge *eb* to get the second polytope $P_1$ in Fig. 5. We move vertex $c$ a small amount so that we get the third polytope $P_2$ in Fig. 5. Finally we rotate edge *eb* a small amount to get the fourth polytope $P_3$ in Fig. 5.

We now construct the sphere $C$ by taking the triangles *adf*, *acf*, *abd*, *abc*, *bed*, *bec*, *cef* and *def*. Note that this sphere is isomorphic to the sphere $C$ in our original diagram, with the isomorphism given by this labeling of vertices.

Now, suppose that $P_1$ was inside another right triangular prism with parallel bases, with vertices $a$, $d$, $h$, $c$, $f$, and $i$, as in Fig. 6. When we moved the vertex $c$ we could have done so keeping $c$ on the two planes determined by *achi* and *cfi*, thus we may assume that $S$ is in the polytope $P' = adfchi$ (Fig. 7).

The polytope $P'$ will soon be modified to become the face $F_1$.

We now will construct the faces $F_2$ and $F_3$ inside this polytope. Let $j$ be a point near the midpoint of edge *eb*, outside $C$ and inside $P'$, such that the edge *ej* is parallel to edge *dh* (and thus parallel to edge *fi*).

Consider the two polytopes *ahbjed* and *cibjef* (Fig. 8). To each we add a pyramidal cap over a quadrilateral face as shown in Fig. 9. These two new vertices are vertices $m$ and $l$, thus we have constructed the facets $F_3$ and $F_2$.

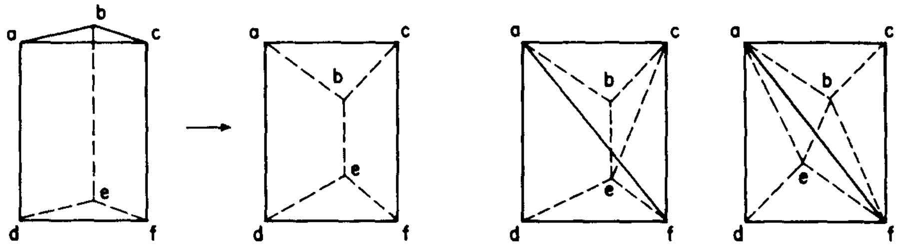


Fig. 5.

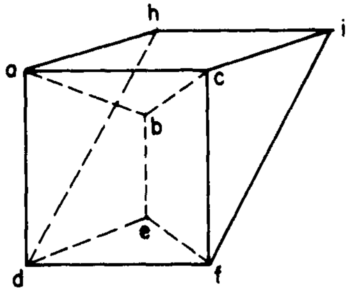


Fig. 6.

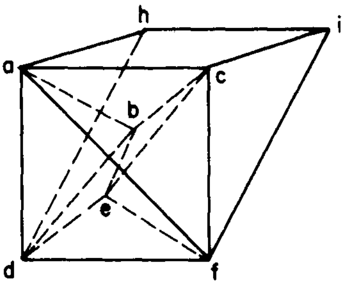


Fig. 7.

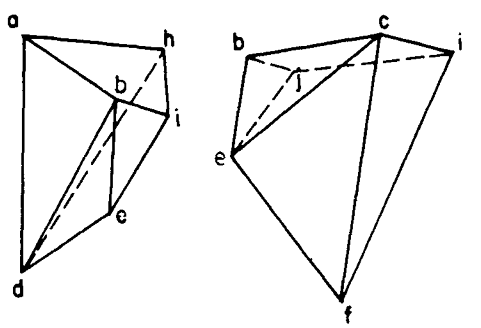


Fig. 8.

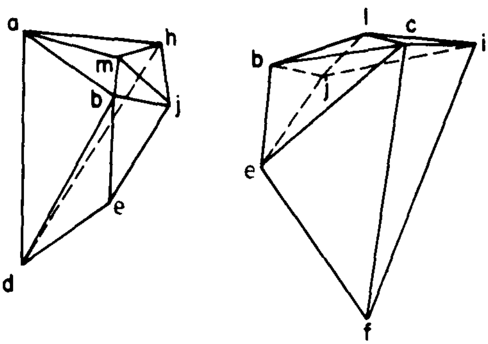


Fig. 9.

If we choose a point $n$ above the planes *abjh* and *cibj*, below the planes *efij* and *dejh*, near $j$ and outside of $F_2$ and $F_3$ then we will have a point such that the segments from $n$ to the vertices $a$, $b$, $c$, $d$, $e$, $f$, $h$ and $i$ will miss the interior of polytopes *abhjed* and *cibjef*, and the interior of $C$. If the vertices $l$ and $m$ were chosen close enough to the polytopes *abhjed* and *cibjef*, then from the point $n$, these two vertices will be visible and all of the vertices $a$, $b$, $c$, $d$, $e$, $f$, $h$ and $l$ will still be visible.

Next we add a pyramidal cap over the face *acih*. The new vertex will be labeled $k$. Clearly, $k$ is visible from $n$. We now add all polytopes of the form $\text{con}(n, X)$ where $X$ is any 2-face visible from $n$.

The construction of the diagram is completed by placing a point $g$ inside $C$, from which all vertices of $C$ are visible, and adding all polytopes of the form $\text{con}(g, X)$ where $X$ is any 2-face of $C$.

Since we have realized the diagram with $F_1$ as the outside facet, it is clear that $F_2$ and $F_3$ can also serve as the outside facet. It remains to be shown that any facet meeting $g$ can serve as the outside facet.

We begin our construction with the toroidal polytope in Fig. 10, with the three triangles *abc*, *def* and *hij* parallel. We rotate the triangle *abc* a small amount about the vertical axis so that we get the toroidal polytope in Fig. 11.

We choose a point $k$ near the midpoint of edge *ch* but outside the polytope *achidf*. The polytope *achidfk* is the facet $F_1$. Similarly we construct the facets $F_2 = cbijfel$, and $F_3 = abdehjm$, by choosing vertices $l$ and $m$ near the midpoints of edges *bi* and *aj*.

We choose a point $n$ at the center of triangle *hij* and add polytopes of the form $\text{con}(n, X)$ where $X$ is any 2-face in the following list: *amh*, *amb*, *mhj*, *bmj*,

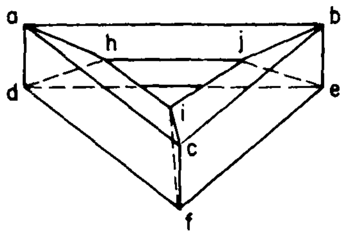


Fig. 10.

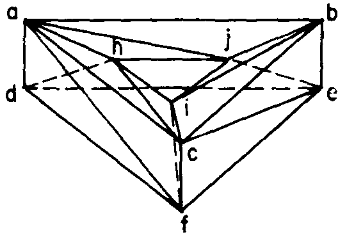


Fig. 11.

*blj*, *lji*, *lic*, *bcl*, *cki*, *kih*, *cka*, *akh*, *abc*, *dfih*, *dejh*, *efij* and *def*. The polytopes we have constructed fill out a polytope $P''$. We complete the construction by choosing a facet $F$ of $P''$., taking a projective transformation of $P''$ such that from some point $g$, all facets of $P''$ but $F$ are visible, then adding all polytopes of the form $\text{con}(g, X)$ where $X$ is a facet of $P''$ visible from $g$.

Since our choice of $F$ was arbitrary, every facet meeting $g$ could serve as the outside facet.

## References

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